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Neutrino Solution of Dirac Equation in Bianchi Type V Cosmological Model

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The Dirac equation in Bianchi type V cosmology is established, and neutrino solutions in the early universe are presented.

1. INTRODUCTION

The discussion of the Dirac equation in curved spaces, particularly in an expanding universe where gravity is believed to play a dominant role in determining the behavior of spin-1/2 particles, is of considerable importance in astrophysics and cosmology. A series of important results has been obtained, most of which have been based upon the FRW metric (Brill and Wheeler, 1957; Isham and Nelson, 1974; Dehnen and Schäfer, 1983; Audretsch and Schäfer, 1978; Chimento and Mollerach, 1986; Barut and Duru, 1987; Kovalyov and Légaré, 1990; Parashar, 1991; Portugal, 1995). This is natural, since the FRW metric is the simplest one. Although the universe seems homogeneous and isotropic at present, there is no observational evidence guaranteeing the isotropy from its very beginning. In fact, there are theoretical arguments that support the existence of an anisotropic phase (Misner, 1968). The Bianchi type I, V, and IX universes are generalizations of FRW models.

Spin is an intrinsic microscopic property of matter. In the early universe, the effective energy density produced by the matter spin could not be negligible, and space may be nonisotropic. Therefore, the theory of the early universe should also consider the effect of the matter spin on space-time geometry,

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and the behavior of relativistic particles obeys the Dirac equation of Einstein–Cartan theory (EC theory) (Hehl *et al.*, 1976). In the recent years, there has been much investigation of cosmological models with spin matter (Nurgaliev and Ponomariev, 1983; Demiański *et al.*, 1986; Gasperini, 1986; Smalley, 1986; Obukhov and Korotky, 1987), and numerical calculations have been performed by the use of different types of Bianchi metrics.

In what follows, we study the Dirac equation of Einstein–Cartan theory in the Bianchi type V cosmological model. In Section 2 we review some useful results in the Bianchi type V cosmological model with spin matter (Lu and Cheng, 1995). Section 3 is devoted to establishing the Dirac equation in Bianchi type V cosmology. In Section 4 the neutrino solutions of the Dirac equation in the early universe are given.

2. REVIEW OF THE BIANCHI TYPE V COSMOLOGICAL MODEL

The line element of the Bianchi type V universe is

$$d\tau^2 = dt^2 - A^2(t)\,dx^2 - B^2(t)e^{-2x}\,dy^2 - C^2(t)e^{-2x}\,dz^2 \tag{2.1}$$

Thus the metric $g_{\mu\nu}(x)$ is given by

$$g_{\mu\nu} = \text{diag}(1, -A^2, -B^2e^{-2x}, -C^2e^{-2x}) \tag{2.2}$$

The Riemann affine connection is

$$\tilde{\Gamma}^{\lambda}_{\nu\mu} = \frac{1}{2} g^{\sigma\lambda}(\partial_\nu g_{\mu\sigma} + \partial_\mu g_{\nu\sigma} - \partial_\sigma g_{\nu\mu}) \tag{2.3}$$

A straightforward but tedious calculation gives the nonvanishing components of the Riemann affine connection as follow:

$$\begin{gathered}
\tilde{\Gamma}^0_{11} = A\dot{A}, \qquad \tilde{\Gamma}^0_{22} = B\dot{B}, \qquad \tilde{\Gamma}^0_{33} = C\dot{C} \\
\tilde{\Gamma}^1_{01} = \tilde{\Gamma}^1_{10} = \frac{\dot{A}}{A}, \qquad \tilde{\Gamma}^1_{22} = \frac{B^2}{A^2}e^{-2x}, \qquad \tilde{\Gamma}^1_{22} = \frac{C^2}{A^2}e^{-2x} \\
\tilde{\Gamma}^2_{02} = \tilde{\Gamma}^2_{20} = \frac{\dot{B}}{B}, \qquad \tilde{\Gamma}^3_{03} = \tilde{\Gamma}^3_{30} = \frac{\dot{C}}{C}, \qquad \tilde{\Gamma}^2_{12} = \tilde{\Gamma}^2_{21} = \tilde{\Gamma}^3_{13} = \tilde{\Gamma}^3_{31} = -1
\end{gathered} \tag{2.4}$$

where the dot denotes d/dt (for example, $\dot{A} = dA/dt$).

In EC theory, the effective Einstein equation is (Obukhov and Korotky, 1987)

$$\tilde{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\tilde{R} = \kappa T^{\text{eff}}_{\mu\nu} \tag{2.5}$$

where

$$T^{\text{eff}}_{\mu\nu} = (\rho_{\text{eff}} + P_{\text{eff}})u_\mu u_\nu - P_{\text{eff}} g_{\mu\nu} - 2(g^{\alpha\beta} + u^\alpha u^\beta)\, \nabla_\alpha [u_{(\mu} S_{\nu)\beta}]$$
$$\rho_{\text{eff}} = \rho - \kappa S^2 = \rho - \frac{1}{2}\kappa S_{\alpha\beta} S^{\alpha\beta} \tag{2.6}$$
$$P_{\text{eff}} = P - \kappa S^2 = P - \frac{1}{2}\kappa S_{\alpha\beta} S^{\alpha\beta}$$

with $\kappa = 8\pi G/c^4$. Here u^μ is the 4-velocity of the fluid element, and $S^{\mu\nu}$ is the spin momentum density, which is a skew-symmetric tensor ($S^{\mu\nu} = -S^{\nu\mu}$) in the Wessenholf model (Weyssenhoff and Raabe, 1947). The conservation law of energy-momentum can be written as

$$u^\mu \nabla_{\mu\rho} + (\rho + P)\nabla_\mu u^\mu = 0 \tag{2.7}$$

The equation of state of a Wessenholf ideal fluid is assumed to be

$$P = \gamma\rho \qquad (0 \leq \gamma \leq 1) \tag{2.8}$$

By use of (2.4), (2.7), and (2.8), Lu and Cheng (1995) solved the effective Einstein equation (2.5) and obtained the solutions for the radiation epoch ($P = 1/3\rho$):

$$A^2 = A_0^2 \operatorname{ch}(2\eta + \delta) - \frac{1}{6} M \tag{2.9}$$

$$B = B_0 A \exp\left[\frac{1}{2} Q \int A^{-2}\, d\eta\right] \tag{2.10}$$

$$C = C_0 A \exp\left[-\frac{1}{2} Q \int A^{-2}\, d\eta\right] \tag{2.11}$$

$$S_{21} = S_{31} = 0 \tag{2.12}$$

where A_0^2, B_0, C_0, δ, Q, and M are constants which satisfy $B_0 C_0 = f$. The value $A_{\min} = \sqrt{A_0^2 - M/6}$ represents the minimum radius of the universe.

3. THE DIRAC EQUATION

To evaluate the spin connection $\tilde{\Gamma}_\mu$, we introduce the vierbein V^a_μ satisfying

$$V^a{}_\mu V_a{}^\nu = \delta^\nu_\mu, \qquad V^a{}_\mu V_b{}^\mu = \delta^a_b \tag{3.1}$$

$$g_{\mu\nu} = V^a{}_\mu V^b{}_\nu \eta_{ab} \tag{3.2}$$

where $\eta_{ab} = \mathrm{diag}(+1, -1, -1, -1)$. One of the solutions of expression (3.2) is

$$V^a{}_\mu = \mathrm{diag}(1, A, Be^{-x}, Ce^{-x}) \tag{3.3}$$

The spin connection is defined by

$$\tilde{\Gamma}_\mu = \frac{1}{2} V_a{}^\nu(\partial_\mu V_{b\nu} - \tilde{\Gamma}^\lambda_{\nu\mu} V_{b\lambda})\sigma^{ab} \tag{3.4}$$

where

$$\sigma^{ab} = \frac{1}{4}[\gamma^a, \gamma^b] \tag{3.5}$$

and γ^a are the Dirac matrices in flat space, which satisfy

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab} \tag{3.6}$$

Substituting (2.4) and (3.3) into (3.4) yields

$$\tilde{\Gamma}_0 = 0, \qquad \tilde{\Gamma}_1 = \frac{1}{2}\dot{A}\gamma^0\gamma^1, \qquad \tilde{\Gamma}_2 = \frac{1}{2}\dot{B}e^{-x}\gamma^0\gamma^2 - \frac{1}{2}\frac{B}{A}e^{-x}\gamma^1\gamma^2$$

$$\tilde{\Gamma}_3 = \frac{1}{2}\dot{C}e^{-x}\gamma^0\gamma^3 - \frac{1}{2}\frac{C}{A}e^{-x}\gamma^1\gamma^3 \tag{3.7}$$

Assuming that the universe is filled with spin matter, then the Dirac equation may be rewritten as (Hehl and Data, 1971)

$$\gamma^\mu\tilde{\nabla}_\mu\psi - \frac{3}{8}il^2(\bar{\psi}\gamma_5\gamma^\mu\psi)\gamma_5\gamma_\mu\psi - im\psi = 0 \tag{3.8}$$

where γ^μ are the Dirac matrices, which satisfy the anticommutation relation

$$\{\gamma^\mu(x), \gamma^\nu(x)\} = 2g^{\mu\nu} \tag{3.9}$$

and $\tilde{\nabla}\psi = (\partial_\mu - \tilde{\Gamma}_\mu)\psi$. The nonliner term $3/8il^2(\bar{\psi}\gamma_5\gamma^\mu\psi)\gamma_5\gamma_\mu\psi$, which represents a spin–spin contact interaction, is an axial vector interaction with characteristic length $l^2 = 8\pi G\hbar/c^3$. Applying expression (3.7) to (3.8) yields the Dirac equation in Bianchi type V cosmology

$$\left[\left(\partial_t + \frac{\dot{A}}{2A} + \frac{\dot{B}}{2B} + \frac{\dot{C}}{2C}\right) + \frac{1}{A}\gamma^0\gamma^1(\partial_x - 1) + \frac{e^x}{B}\gamma^0\gamma^2\partial_y \right.$$
$$\left. + \frac{e^x}{C}\gamma^0\gamma^3\partial_z - \frac{3}{2}ia^\mu\gamma^0\gamma_5\gamma_\mu - im\right]\psi = 0 \tag{3.10}$$

where $a^\mu = 1/4l^2(\bar{\psi}\gamma_5\gamma^\mu\psi)$ is the axial-vector part of torsion, which may be regarded as the background torsion generated by the spin of particles distributed over all the universe with mass μ and number density $n(n = \rho/\mu)$. Equation (2.11) shows that the direction of matter spin is along the positive x axis. The background wave function may be taken as

$$\psi = u = u(0)e^{iEt}, \qquad u(0) = \sqrt{\frac{n}{2}}\,(1\ 1\ 0\ 0)^T \tag{3.11}$$

Then Dirac equation (3.10) may be simplified to

$$\left[\left(\partial_t + \frac{\dot{A}}{2A} + \frac{\dot{B}}{2B} + \frac{\dot{C}}{2C}\right) + \frac{1}{A}\alpha^1(\partial_x - 1) + \frac{e^x}{B}\alpha^2\partial_y + \frac{e^x}{C}\alpha^3\partial_z - \frac{3}{8}il^2\frac{\rho}{\mu}\alpha + i\gamma^0 m\right]\psi = 0 \tag{3.12}$$

where

$$\alpha^1 = \gamma^0\gamma^1, \qquad \alpha^2 = \gamma^0\gamma^2, \qquad \alpha^3 = \gamma^0\gamma^3, \qquad \alpha = (\bar{u}\gamma_5\gamma^\mu u)\gamma_5\gamma_\mu \tag{3.13}$$

Taking the solution in the form of

$$\psi = \frac{1}{(ABC)^{1/2}} f(t)e^{ikx} \tag{3.14}$$

we find that (3.12) reduces to

$$\left[\partial_t + \frac{\alpha^1}{A}(ik - 1) + i\beta - im\gamma^0\right]f(t) = 0 \tag{3.15}$$

where $\beta = 3\hbar cM/8\mu A^4 = \lambda A^{-4}$. Equation (3.15) is equivalent to

$$\begin{pmatrix} \partial_t + i(\beta - m) & 0 & 0 & \dfrac{ik-1}{A} \\ 0 & \partial_t + i(\beta - m) & \dfrac{ik-1}{A} & 0 \\ 0 & \dfrac{ik-1}{A} & \partial_t + i(\beta + m) & 0 \\ \dfrac{ik-1}{A} & 0 & 0 & \partial_t + i(\beta + m) \end{pmatrix} \times \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = 0 \tag{3.16}$$

The components f_1 and f_2 obey the differential equations

$$\left\{\partial_t^2 + \left[\frac{\dot{A}}{A} + i(\beta + m)\right]\partial_t + i\dot{\beta} + \frac{\dot{A}}{A} i(\beta + m) - (\beta^2 - m^2) - \left(\frac{ik - 1}{A}\right)^2\right\} f_{1,2} = 0 \tag{3.17}$$

while the components f_3 and f_4 satisfy

$$f_{4,3} = \frac{A}{1 - ik}[\partial_t + i(\beta - m)]f_{1,2} \tag{3.18}$$

Equations (3.17) and (3.18) may be rewritten in the form

$$\left[\partial_\eta^2 + iA(\beta + m)\partial_\eta + iA'(\frac{3}{4}\beta - m) - A^2(\beta^2 - m^2) - (ik - 1)^2\right] f_{1,2} = 0 \tag{3.19}$$

$$f_{4,3} = \frac{1}{1 - ik}[\partial_\eta + iA(\beta - m)]\, f_{1,2} \tag{3.20}$$

where the prime denotes $d/d\eta \equiv A\, d/dt$.

4. THE NEUTRINO SOLUTION OF THE DIRAC EQUATION IN THE EARLY UNIVERSE

For simplicity, we consider only the condition of the Dirac equation for the neutrino ($m = 0$). Then equations (3.19) and (3.20) reduce to

$$\left[\partial_\eta^2 + \frac{i\lambda}{A^3}\partial_\eta + \frac{3}{4} i\lambda \frac{A'}{A^4} - \frac{\lambda^2}{A^6} - (ik - 1)^2\right] f_{1,2} = 0 \tag{4.1}$$

$$f_{4,3} = \frac{1}{1 - ik}[\partial_\eta + i\lambda A^{-3}]\, f_{1,2} \tag{4.2}$$

If we choose

$$u(\eta) = f(\eta) \exp\left[-\frac{1}{2}\int i\lambda A^{-3}\, d\eta\right] \tag{4.3}$$

then equation (4.1) becomes

$$\frac{d^2u(\eta)}{d\eta^2} + I(\eta)u(\eta) = 0 \tag{4.4}$$

where

$$I(\eta) = -\frac{3}{4}\lambda^2 A^{-6} + \frac{9}{4} i\lambda A' A^{-4} - (ik - 1)^2 \tag{4.5}$$

is an invariant of equation (4.1). As η is very small in the early universe, we may expand $I(\eta)$ to the second power of η:

$$I(\eta) = D + E\eta + F\eta^2 + O(\eta^3) \tag{4.6}$$

where

$$D = -\frac{3}{4}\lambda^2\left(A_0^2 \operatorname{ch}\delta - \frac{M}{6}\right)^{-3} + \frac{9}{4} i\lambda A_0^2\left(A_0^2 \operatorname{ch}\delta - \frac{M}{6}\right)^{-5/2} \operatorname{sh}\delta - (ik - 1)^2 \tag{4.7}$$

$$E = \frac{9}{4} i\lambda\left[2A_0^2\left(A_0^2 \operatorname{ch}\delta - \frac{M}{6}\right)^{-5/2} \operatorname{ch}\delta - 5A_0^4\left(A_0^2 \operatorname{ch}\delta - \frac{M}{6}\right)^{-7/2} \operatorname{sh}^2\delta - 2i\lambda A_0^2\left(A_0^2 \operatorname{ch}\delta - \frac{M}{6}\right)^{-4} \operatorname{sh}\delta\right] \tag{4.8}$$

$$F = -\frac{3}{2}\lambda^2\left[12A_0^2\left(A_0^2 \operatorname{ch}\delta - \frac{M}{6}\right)^{-5} \operatorname{sh}^2\delta - 3A_0^2\left(A_0^2 \operatorname{ch}\delta - \frac{M}{6}\right)^{-4} \operatorname{ch}\delta\right] + \frac{9}{8} i\lambda\left[4A_0^2\left(A_0^2 \operatorname{ch}\delta - \frac{M}{6}\right)^{-5/2} \operatorname{sh}\delta - 30A_0^4\left(A_0^2 \operatorname{ch}\delta - \frac{M}{6}\right)^{-7/2} \operatorname{sh}\delta \operatorname{ch}\delta + 35A_0^6 \operatorname{sh}^3\delta\left(A_0^2\, ch\,\delta - \frac{M}{6}\right)^{-9/2}\right] \tag{4.9}$$

Thus equation (4.4) becomes

$$\frac{d^2u(\eta)}{d\eta^2} + (D + E\eta + F\eta^2)u(\eta) = 0 \tag{4.10}$$

Equation (4.10) can be reduced to the form

$$\frac{d^2v(x)}{dx^2} + (a + bx^2)v(x) = 0 \tag{4.11}$$

with

$$a = D - \frac{E^2}{4F}, \qquad b = F \tag{4.12}$$

by the use of substitution of variables

$$u(\eta) = v(x), \qquad 2F(x - \eta) = E \tag{4.13}$$

Let

$$v(x) = w(x), \qquad cx^2 = z^2 \tag{4.14}$$

Equation (4.11) can be further reduced to

$$\frac{d^2w(z)}{dz^2} + \left(e + \frac{1}{2} - \frac{z^2}{4}\right)w(z) = 0 \tag{4.15}$$

where

$$\frac{a}{c} = 2e + 1, \qquad \frac{b}{c^2} + \frac{1}{4} = 0 \tag{4.16}$$

Equation (4.15) is Weber's equation (Murphy, 1960). Its solutions are parabolic cylinder or Weber–Hermite functions. One of the solutions is

$$D_e(z) = 2^{e/2+1/4}z^{-1/2}W_{(e/2+1/4),-1/4}\left(\frac{z^2}{2}\right) \tag{4.17}$$

where $W_{(e/2+1/4),-1/4}(z^2/2)$ is the Whittaker function. The other solution is $D_{-(e+1)}(iz)$. Thus we get

$$f_{1,2}(\eta) = \left\{C_1D_e\left[\sqrt{c}\left(\eta + \frac{E}{2F}\right)\right] + C_2D_{e+1}\left[i\sqrt{c}\left(\eta + \frac{E}{2F}\right)\right]\right\} \exp\left[\frac{1}{2}\int i\lambda A^{-3}\, d\eta\right] \tag{4.18}$$

$$f_{4,3}(\eta) = \frac{\exp\left[\frac{1}{2}\int i\lambda A^{-3}\,d\eta\right]}{1 - ik}$$
$$\times \left(C_1\left\{eD_{e-1}\left[\sqrt{c}\left(\eta + \frac{E}{2F}\right)\right]\right.\right.$$
$$\left. + \left[\frac{3i\lambda A^{-3}}{2} - \frac{\sqrt{c}}{2}\left(\eta + \frac{E}{2F}\right)\right]D_e\left[\sqrt{c}\left(\eta + \frac{E}{2F}\right)\right]\right\}$$
$$+ C_2\left\{-(e + 1)D_{-e-2}\left[i\sqrt{c}\left(\eta + \frac{E}{2F}\right)\right]\right.$$
$$\left.\left. + \left[\frac{3i\lambda A^{-3}}{2} - \frac{i\sqrt{c}}{2}\left(\eta + \frac{E}{2F}\right)\right]D_{-e-1}\left[i\sqrt{c}\left(\eta + \frac{E}{2F}\right)\right]\right\}\right) \tag{4.19}$$

Then the four independent solutions of the form (3.14) are

$$\psi_1 = f^{-1/2}\left[A_0\,\mathrm{ch}(2\eta + \delta) - \frac{M}{6}\right]^{-3/2}\exp\left[ikx + \frac{1}{2}\int i\lambda A^{-3}\,d\eta\right]$$
$$\times \begin{pmatrix} D_e\left[\sqrt{c}\left(\eta + \frac{E}{2F}\right)\right] \\ 0 \\ 0 \\ \frac{1}{1 - ik}\left\{eD_{e-1}\left[\sqrt{c}\left(\eta + \frac{E}{2F}\right)\right] + \left[\frac{3}{2}i\lambda A^{-3} - \frac{\sqrt{c}}{2}\left(\eta + \frac{E}{2F}\right)\right]D_e\left[\sqrt{c}\left(\eta + \frac{E}{2F}\right)\right]\right\} \end{pmatrix} \tag{4.20}$$

$$\psi_2 = f^{-1/2}\left[A_0\,\mathrm{ch}(2\eta + \delta) - \frac{M}{6}\right]^{-3/2}\exp\left[ikx + \frac{1}{2}\int i\lambda A^{-3}\,d\eta\right]$$
$$\times \begin{pmatrix} D_{-e-1}\left[i\sqrt{c}\left(\eta + \frac{E}{2F}\right)\right] \\ 0 \\ 0 \\ \frac{1}{1 - ik}\left\{-(e + 1)D_{-e-2}\left[i\sqrt{c}\left(\eta + \frac{E}{2F}\right)\right] + \left[\frac{3}{2}i\lambda A^{-3} - \frac{i\sqrt{c}}{2}\left(\eta + \frac{E}{2F}\right)\right]D_{-e-1}\left[i\sqrt{c}\left(\eta + \frac{E}{2F}\right)\right]\right\} \end{pmatrix} \tag{4.21}$$

$$\psi_3 = f^{-1/2}\left[A_0 \,\mathrm{ch}(2\eta + \delta) - \frac{M}{6}\right]^{-3/2} \exp\left[ikx + \frac{1}{2}\int i\lambda A^{-3}\, d\eta\right]$$
$$\times \begin{pmatrix} 0 \\ D_e\left[\sqrt{c}\left(\eta + \frac{E}{2F}\right)\right] \\ \frac{1}{1 - ik}\left\{eD_{e-1}\left[\sqrt{c}\left(\eta + \frac{E}{2F}\right)\right] \right. \\ \left. + \left[\frac{3}{2} i\lambda A^{-3} - \frac{\sqrt{c}}{2}\left(\eta + \frac{E}{2F}\right)\right] D_e\left[\sqrt{c}\left(\eta + \frac{E}{2F}\right)\right]\right\} \\ 0 \end{pmatrix} \tag{4.22}$$

$$\psi_4 = f^{-1/2}\left[A_0 \,\mathrm{ch}(2\eta + \delta) - \frac{M}{6}\right]^{-3/2} \exp\left[ikx + \frac{1}{2}\int i\lambda A^{-3}\, d\eta\right]$$
$$\times \begin{pmatrix} 0 \\ D_{-e-1}\left[i\sqrt{c}\left(\eta + \frac{E}{2F}\right)\right] \\ \frac{1}{1 - ik}\left\{-(e + 1)D_{-e-2}\left[i\sqrt{c}\left(\eta + \frac{E}{2F}\right)\right] \right. \\ \left. + \left[\frac{3}{2} i\lambda A^{-3} - \frac{i\sqrt{c}}{2}\left(\eta + \frac{E}{2F}\right)\right] D_{-e-1}\left[i\sqrt{c}\left(\eta + \frac{E}{2F}\right)\right]\right\} \\ 0 \end{pmatrix} \tag{4.23}$$

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